We begin by simplifying

$$\frac{a+bi}{b+ai} = \frac{a+bi}{b+ai} \times \frac{b-ai}{b-ai} = \frac{2ab}{a^2+b^2} + \frac{b^2-a^2}{a^2+b^2} i. \qquad (1.1.23)$$

Therefore,

$$\left|\frac{a+bi}{b+ai}\right| = \sqrt{\frac{4a^2b^2}{(a^2+b^2)^2} + \frac{b^4-2a^2b^2+a^4}{(a^2+b^2)^2}} = \sqrt{\frac{a^4+2a^2b^2+b^4}{(a^2+b^2)^2}} = 1. \qquad (1.1.24)$$

Problems

Simplify the following complex numbers. Represent the solution in the Cartesian form $a + bi$:

1. $\dfrac{5i}{2+i}$

2. $\dfrac{5+5i}{3-4i} + \dfrac{20}{4+3i}$

3. $\dfrac{1+2i}{3-4i} + \dfrac{2-i}{5i}$

4. $(1-i)^4$

5. $i(1-i\sqrt{3})(\sqrt{3}+i)$

Represent the following complex numbers in polar form:

6. $-i$

7. -4

8. $2+2\sqrt{3}\,i$

9. $-5+5i$

10. $2-2i$

11. $-1+\sqrt{3}\,i$

12. By the law of exponents, $e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$. Use Euler's formula to obtain expressions for $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$ in terms of sines and cosines of α and β.

13. Using the property that $\sum_{n=0}^{N} q^n = (1-q^{N+1})/(1-q)$ and the geometric series $\sum_{n=0}^{N} e^{int}$, obtain the following sums of trigonometric functions:

$$\sum_{n=0}^{N} \cos(nt) = \cos\left(\frac{Nt}{2}\right) \frac{\sin[(N+1)t/2]}{\sin(t/2)}$$

and

$$\sum_{n=1}^{N} \sin(nt) = \sin\left(\frac{Nt}{2}\right) \frac{\sin[(N+1)t/2]}{\sin(t/2)}.$$

These results are often called *Lagrange's trigonometric identities.*

14. (a) Using the property that $\sum_{n=0}^{\infty} q^n = 1/(1-q)$, if $|q| < 1$, and the geometric series $\sum_{n=0}^{\infty} \epsilon^n e^{int}$, $|\epsilon| < 1$, show that

$$\sum_{n=0}^{\infty} \epsilon^n \cos(nt) = \frac{1 - \epsilon \cos(t)}{1 + \epsilon^2 - 2\epsilon \cos(t)}$$

and

$$\sum_{n=1}^{\infty} \epsilon^n \sin(nt) = \frac{\epsilon \sin(t)}{1 + \epsilon^2 - 2\epsilon \cos(t)}.$$

(b) Let $\epsilon = e^{-a}$, where $a > 0$. Show that

$$2\sum_{n=1}^{\infty} e^{-na} \sin(nt) = \frac{\sin(t)}{\cosh(a) - \cos(t)}.$$

1.2 FINDING ROOTS

The concept of finding roots of a number, which is rather straightforward in the case of real numbers, becomes more difficult in the case of complex numbers. By finding the *roots* of a complex number, we wish to find all the solutions w of the equation $w^n = z$, where n is a positive integer for a given z.

We begin by writing z in the polar form:

$$z = re^{i\varphi} \tag{1.2.1}$$

while we write

$$w = Re^{i\Phi} \tag{1.2.2}$$

for the unknown. Consequently,

$$w^n = R^n e^{in\Phi} = re^{i\varphi} = z. \tag{1.2.3}$$

We satisfy (1.2.3) if

$$R^n = r \quad \text{and} \quad n\Phi = \varphi + 2k\pi, \quad k = 0, \pm1, \pm2, \ldots, \tag{1.2.4}$$

because the addition of any multiple of 2π to the argument is also a solution. Thus, $R = r^{1/n}$, where R is the uniquely determined real positive root, and

$$\Phi_k = \frac{\varphi}{n} + \frac{2\pi k}{n}, \qquad k = 0, \pm1, \pm2, \ldots \tag{1.2.5}$$

Because $-1+i=\sqrt{2}\exp(3\pi i/4)$,

$$z_k = 2^{1/6}\exp\left(\frac{\pi i}{4}+\frac{2i\pi k}{3}\right), \qquad k=0,1,2, \tag{1.2.14}$$

or

$$z_0 = 2^{1/6}\exp\left(\frac{\pi i}{4}\right) = 2^{1/6}\left[\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right], \tag{1.2.15}$$

$$z_1 = 2^{1/6}\exp\left(\frac{11\pi i}{12}\right) = 2^{1/6}\left[\cos\left(\frac{11\pi}{12}\right)+i\sin\left(\frac{11\pi}{12}\right)\right] \tag{1.2.16}$$

and

$$z_2 = 2^{1/6}\exp\left(\frac{19\pi i}{12}\right) = 2^{1/6}\left[\cos\left(\frac{19\pi}{12}\right)+i\sin\left(\frac{19\pi}{12}\right)\right]. \tag{1.2.17}$$

Figure 1.2.2 gives the location of these zeros on the complex plane.

Problems

Extract all of the possible roots of the following complex numbers:

1. $8^{1/6}$
2. $(-1)^{1/3}$
3. $(-i)^{1/3}$
4. $(-27i)^{1/6}$

5. Find algebraic expressions for the square roots of $a-bi$, where $a>0$ and $b>0$.

6. Find all of the roots for the algebraic equation $z^4-3iz^2-2=0$.

7. Find all of the roots for the algebraic equation $z^4+6iz^2+16=0$.

1.3 THE DERIVATIVE IN THE COMPLEX PLANE: THE CAUCHY-RIEMANN EQUATIONS

In the previous two sections, we have done complex arithmetic. We are now ready to introduce the concept of function as it applies to complex variables.

We have already introduced the complex variable $z=x+iy$, where x and y are variable. We now define another complex variable $w=u+iv$ so that for each value of z there corresponds a value of $w=f(z)$. From all of the possible complex functions that we might invent, we will focus on those functions where for each z there is one, and only one, value of w. These functions are *single-valued*. They differ from functions such

Problems

Show that the following functions are entire:

1. $f(z) = iz + 2$
2. $f(z) = e^{-z}$
3. $f(z) = z^3$
4. $f(z) = \cosh(z)$

Find the derivative of the following functions:

5. $f(z) = (1 + z^2)^{3/2}$
6. $f(z) = (z + 2z^{1/2})^{1/3}$
7. $f(z) = (1 + 4i)z^2 - 3z - 2$
8. $f(z) = (2z - i)/(z + 2i)$
9. $f(z) = (iz - 1)^{-3}$

Evaluate the following limits:

10. $\lim_{z \to i} \frac{z^2 - 2iz - 1}{z^4 + 2z^2 + 1}$
11. $\lim_{z \to 0} \frac{z - \sin(z)}{z^3}$

12. Show that the function $f(z) = z^*$ is nowhere differentiable.

For each of the following $u(x, y)$, show that it is harmonic and then find a corresponding $v(x, y)$ such that $f(z) = u + iv$ is analytic.

13.
$$u(x, y) = x^2 - y^2$$

14.
$$u(x, y) = x^4 - 6x^2y^2 + y^4 + x$$

15.
$$u(x, y) = x\cos(x)e^{-y} - y\sin(x)e^{-y}$$

16.
$$u(x, y) = (x^2 - y^2)\cos(y)e^x - 2xy\sin(y)e^x$$

1.4 LINE INTEGRALS

So far, we discussed complex numbers, complex functions, and complex differentiation. We are now ready for integration.

Just as we have integrals involving real variables, we can define an integral that involves complex variables. Because the z-plane is two-dimensional there is clearly greater freedom in what we mean by a complex integral. For example, we might ask whether the integral of some function between points A and B depends upon the curve along which

From Figure 1.4.3,

$$\oint_C z\,dz = \int_{C_1} z\,dz + \int_{C_2} z\,dz + \int_{C_3} z\,dz. \tag{1.4.18}$$

Now

$$\int_{C_1} z\,dz = \int_1^0 iy\,(i\,dy) = -\int_1^0 y\,dy = -\left.\frac{y^2}{2}\right|_1^0 = \frac{1}{2}, \tag{1.4.19}$$

$$\int_{C_2} z\,dz = \int_0^{-1} x\,dx = \left.\frac{x^2}{2}\right|_0^{-1} = \frac{1}{2} \tag{1.4.20}$$

and

$$\int_{C_3} z\,dz = \int_{-\pi}^{\pi/2} e^{\theta i} ie^{\theta i}\,d\theta = \left.\frac{e^{2\theta i}}{2}\right|_{-\pi}^{\pi/2} = -1, \tag{1.4.21}$$

where we have used $z = e^{\theta i}$ around the portion of the unit circle. Therefore, the closed line integral equals zero.

• Example 1.4.4

Let us integrate $f(z) = 1/(z-a)$ around any circle centered on $z = a$. The Cauchy-Riemann equations show that $f(z)$ is a meromorphic function. It is analytic everywhere except at the isolated singularity $z = a$.

If we introduce polar coordinates by letting $z - a = re^{\theta i}$ and $dz = ire^{\theta i}\,d\theta$,

$$\oint_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{ire^{\theta i}}{re^{\theta i}}\,d\theta = i\int_0^{2\pi} d\theta = 2\pi i. \tag{1.4.22}$$

Note that the integrand becomes undefined at $z = a$. Furthermore, the answer is independent of the size of the circle. Our example suggests that when we have a closed contour integration it is the behavior of the function within the contour rather than the exact shape of the closed contour that is of importance. We will return to this point in later sections.

Problems

1. Evaluate $\oint_C (z^*)^2\,dz$ around the circle $|z| = 1$ taken in the counterclockwise direction.

2. Evaluate $\oint_C |z|^2\,dz$ around the square with vertices at (0,0), (1,0), (1,1), and (0,1) taken in the counterclockwise direction.

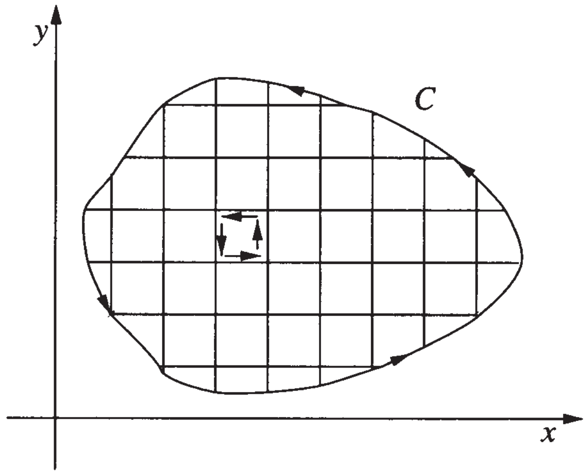

Figure 1.5.1: Diagram used in proving the Cauchy-Goursat theorem.

3. Evaluate $\int_C |z|\,dz$ along the right half of the circle $|z| = 1$ from $z = -i$ to $z = i$.

4. Evaluate $\int_C e^z\,dz$ along the line $y = x$ from $(-1, -1)$ to $(1, 1)$.

5. Evaluate $\int_C (z^*)^2\,dz$ along the line $y = x^2$ from $(0, 0)$ to $(1, 1)$.

6. Evaluate $\int_C z^{-1/2}\,dz$, where C is (a) the upper semicircle $|z| = 1$ and (b) the lower semicircle $|z| = 1$. If $z = re^{\theta i}$, restrict $-\pi < \theta < \pi$. Take both contours in the counterclockwise direction.

1.5 THE CAUCHY-GOURSAT THEOREM

In the previous section we showed how to evaluate line integrations by brute-force reduction to real-valued integrals. In general, this direct approach is quite difficult and we would like to apply some of the deeper properties of complex analysis to work smarter. In the remaining portions of this chapter we will introduce several theorems that will do just that.

If we scan over the examples worked in the previous section, we see considerable differences when the function was analytic inside and on the contour and when it was not. We may formalize this anecdotal evidence into the following theorem:

Cauchy-Goursat theorem[2]: *Let $f(z)$ be analytic in a domain D and*

[2] See Goursat, E., 1900: Sur la définition générale des fonctions analytiques, d'après Cauchy. *Trans. Am. Math. Soc.*, **1**, 14–16.

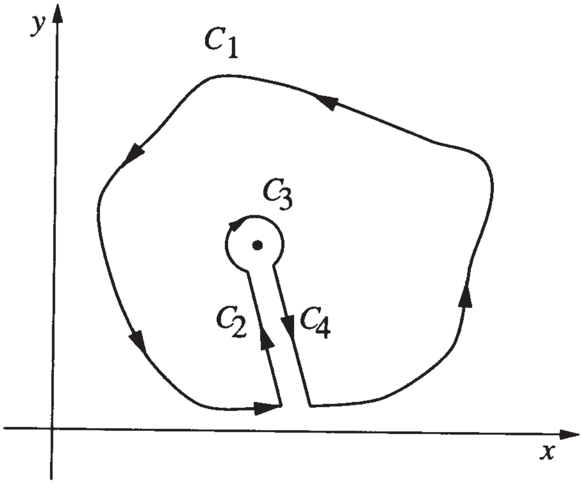

Figure 1.6.1: Diagram used to prove Cauchy's integral formula.

Problems

For the following integrals, show that they are path independent and determine the value of the integral:

1. $\int_{1-\pi i}^{2+3\pi i} e^{-2z}\, dz$
2. $\int_{0}^{2\pi} [e^z - \cos(z)]\, dz$
3. $\int_{0}^{\pi} \sin^2(z)\, dz$
4. $\int_{-i}^{2i} (z+1)\, dz$

1.6 CAUCHY'S INTEGRAL FORMULA

In the previous section, our examples suggested that the presence of a singularity within a contour really determines the value of a closed contour integral. Continuing with this idea, let us consider a class of closed contour integrals that explicitly contain a single singularity within the contour, namely $\oint_C g(z)\, dz$, where $g(z) = f(z)/(z - z_0)$ and $f(z)$ is analytic within and on the contour C. We have closed the contour in the *positive sense* where the enclosed area lies to your left as you move along the contour.

We begin by examining a closed contour integral where the closed contour consists of the C_1, C_2, C_3, and C_4 as shown in Figure 1.6.1. The gap or cut between C_2 and C_4 is very small. Because $g(z)$ is analytic within and on the closed integral, we have that

$$\int_{C_1} \frac{f(z)}{z - z_0}\, dz + \int_{C_2} \frac{f(z)}{z - z_0}\, dz + \int_{C_3} \frac{f(z)}{z - z_0}\, dz + \int_{C_4} \frac{f(z)}{z - z_0}\, dz = 0. \tag{1.6.1}$$

Therefore, $f(z) = e^z/(z-3)$, $n = 1$ and $z_0 = 1$. The function $f(z)$ is analytic within the closed contour because the point $z_0 = 3$ lies outside of the contour. Applying Cauchy's integral formula,

$$\oint_{|z|=2} \frac{e^z}{(z-1)^2(z-3)}\,dz = \frac{2\pi i}{1!}\frac{d}{dz}\left(\frac{e^z}{z-3}\right)\Bigg|_{z=1} \tag{1.6.16}$$

$$= 2\pi i\left[\frac{e^z}{z-3} - \frac{e^z}{(z-3)^2}\right]\Bigg|_{z=1} \tag{1.6.17}$$

$$= -\frac{3\pi i e}{2}. \tag{1.6.18}$$

Problems

Use Cauchy's integral formula to evaluate the following integrals. Assume all of the contours are in the positive sense.

1. $\displaystyle\oint_{|z|=1} \frac{\sin^6(z)}{z-\pi/6}\,dz$

2. $\displaystyle\oint_{|z|=1} \frac{\sin^6(z)}{(z-\pi/6)^3}\,dz$

3. $\displaystyle\oint_{|z|=1} \frac{1}{z(z^2+4)}\,dz$

4. $\displaystyle\oint_{|z|=1} \frac{\tan(z)}{z}\,dz$

5. $\displaystyle\oint_{|z-1|=1/2} \frac{1}{(z-1)(z-2)}\,dz$

6. $\displaystyle\oint_{|z|=5} \frac{\exp(z^2)}{z^3}\,dz$

7. $\displaystyle\oint_{|z-1|=1} \frac{z^2+1}{z^2-1}\,dz$

8. $\displaystyle\oint_{|z|=2} \frac{z^2}{(z-1)^4}\,dz$

9. $\displaystyle\oint_{|z|=2} \frac{z^3}{(z+i)^3}\,dz$

10. $\displaystyle\oint_{|z|=1} \frac{\cos(z)}{z^{2n+1}}\,dz$

1.7 TAYLOR AND LAURENT EXPANSIONS AND SINGULARITIES

In the previous section we showed what a crucial role singularities play in complex integration. Before we can find the most general way of computing a closed complex integral, our understanding of singularities must deepen. For this, we employ power series.

One reason why power series are so important is their ability to provide locally a general representation of a function even when its arguments are complex. For example, when we were introduced to trigonometric functions in high school, it was in the context of a right triangle and a real angle. However, when the argument becomes complex this geometrical description disappears and power series provide a formalism for defining the trigonometric functions, regardless of the nature of the argument.

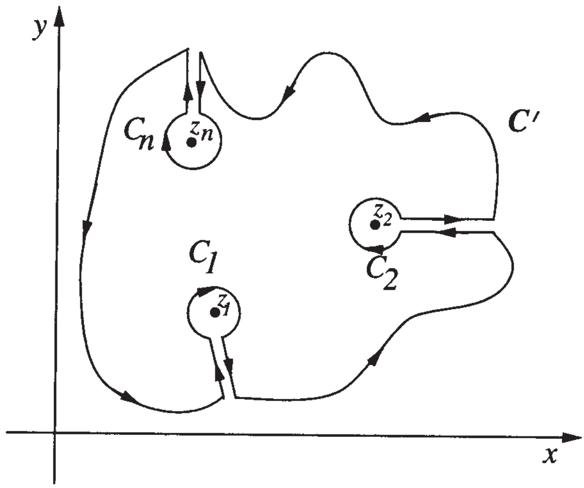

Figure 1.8.1: Contour used in deriving the residue theorem.

Problems

1. Find the Taylor expansion of $f(z) = (1-z)^{-2}$ about the point $z = 0$.

2. Find the Taylor expansion of $f(z) = (z-1)e^z$ about the point $z = 1$. [Hint: Don't find the expansion by taking derivatives.]

By constructing a Laurent expansion, describe the type of singularity and give the residue at z_0 for each of the following functions:

3. $f(z) = z^{10}e^{-1/z}; \quad z_0 = 0$

4. $f(z) = z^{-3}\sin^2(z); \quad z_0 = 0$

5. $f(z) = \dfrac{\cosh(z) - 1}{z^2}; \quad z_0 = 0$

6. $f(z) = \dfrac{z}{(z+2)^2}; \quad z_0 = -2$

7. $f(z) = \dfrac{e^z + 1}{e^{-z} - 1}; \quad z_0 = 0$

8. $f(z) = \dfrac{e^{iz}}{z^2 + b^2}; \quad z_0 = bi$

9. $f(z) = \dfrac{1}{z(z-2)}; \quad z_0 = 2$

10. $f(z) = \dfrac{\exp(z^2)}{z^4}; \quad z_0 = 0$

1.8 THEORY OF RESIDUES

Having shown that around any singularity we may construct a Laurent expansion, we now use this result in the integration of closed complex integrals. Consider a closed contour in which the function $f(z)$ has a number of isolated singularities. As we did in the case of Cauchy's integral formula, we introduce a new contour C' which excludes all of the singularities because they are isolated. See Figure 1.8.1. Therefore, Similarly, the residue at $z = -1 - i$ is

$$\operatorname{Res}\left[\frac{e^{tz}}{z^2(z^2+2z+2)}; -1-i\right] = \lim_{z\to -1-i} [z-(-1-i)]\frac{e^{tz}}{z^2(z^2+2z+2)} \tag{1.8.24}$$

$$= \left(\lim_{z\to -1-i} \frac{e^{tz}}{z^2}\right)\left(\lim_{z\to -1-i} \frac{z+1+i}{z^2+2z+2}\right) \tag{1.8.25}$$

$$= \frac{\exp[(-1-i)t]}{2i(-1-i)^2} = \frac{\exp[(-1-i)t]}{4}. \tag{1.8.26}$$

Then by the residue theorem,

$$\frac{1}{2\pi i}\oint_C \frac{e^{tz}}{z^2(z^2+2z+2)}\,dz = \operatorname{Res}\left[\frac{e^{tz}}{z^2(z^2+2z+2)}; 0\right] + \operatorname{Res}\left[\frac{e^{tz}}{z^2(z^2+2z+2)}; -1+i\right] + \operatorname{Res}\left[\frac{e^{tz}}{z^2(z^2+2z+2)}; -1-i\right] \tag{1.8.27}$$

$$= \frac{t-1}{2} + \frac{\exp[(-1+i)t]}{4} + \frac{\exp[(-1-i)t]}{4} \tag{1.8.28}$$

$$= \tfrac{1}{2}\left[t - 1 + e^{-t}\cos(t)\right]. \tag{1.8.29}$$

Problems

Assuming that all of the following closed contours are in the positive sense, use the residue theorem to evaluate the following integrals:

1. $\displaystyle\oint_{|z|=1} \frac{z+1}{z^4-2z^3}\,dz$

2. $\displaystyle\oint_{|z|=1} \frac{(z+4)^3}{z^4+5z^3+6z^2}\,dz$

3. $\displaystyle\oint_{|z|=1} \frac{1}{1-e^z}\,dz$

4. $\displaystyle\oint_{|z|=2} \frac{z^2-4}{(z-1)^4}\,dz$

5. $\displaystyle\oint_{|z|=2} \frac{z^3}{z^4-1}\,dz$

6. $\displaystyle\oint_{|z|=1} z^n e^{2/z}\,dz, \quad n>0$

7. $\displaystyle\oint_{|z|=1} e^{1/z}\cos(1/z)\,dz$

8. $\displaystyle\oint_{|z|=2} \frac{2+4\cos(\pi z)}{z(z-1)^2}\,dz$

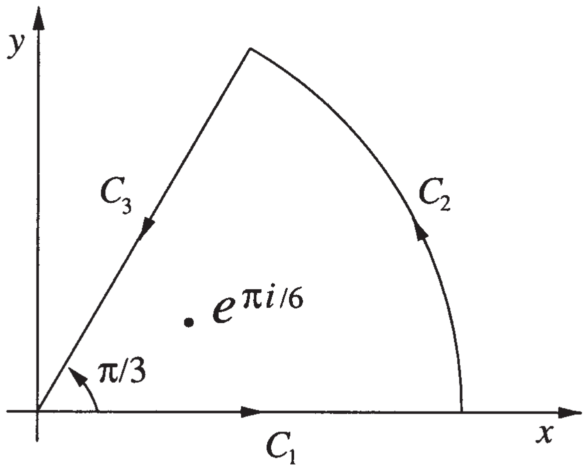

Figure 1.9.2: Contour used in evaluating the integral (1.9.13).

since $z = re^{\pi i/3}$.

Substituting into (1.9.15),

$$\left(1 - e^{\pi i/3}\right) \int_0^\infty \frac{dx}{x^6+1} = -\frac{\pi i}{3} e^{\pi i/6} \tag{1.9.19}$$

or

$$\int_0^\infty \frac{dx}{x^6+1} = \frac{\pi i}{6} \frac{2ie^{\pi i/6}}{e^{\pi i/6}\left(e^{\pi i/6} - e^{-\pi i/6}\right)} = \frac{\pi}{6\sin(\pi/6)} = \frac{\pi}{3}. \tag{1.9.20}$$

Problems

Use the residue theorem to verify the following integral:

1.

$$\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi\sqrt{2}}{4}$$

2.

$$\int_{-\infty}^\infty \frac{dx}{(x^2+4x+5)^2} = \frac{\pi}{2}$$

3.

$$\int_{-\infty}^\infty \frac{x\,dx}{(x^2+1)(x^2+2x+2)} = -\frac{\pi}{5}$$

4.

$$\int_0^\infty \frac{x^2}{x^6+1}\,dx = \frac{\pi}{6}$$

5.

$$\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$$

6.

$$\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)^2} = \frac{5\pi}{288}$$

7.

$$\int_0^\infty \frac{t^2}{(t^2+1)[t^2(a/h+1)+(a/h-1)]}\,dt = \frac{\pi}{4}\left[1-\sqrt{\frac{1-h/a}{1+h/a}}\right],$$

where $h/a < 1$.

8. During an electromagnetic calculation, Strutt[8] needed to prove that

$$\pi\frac{\sinh(\sigma x)}{\cosh(\sigma\pi)} = 2\sigma\sum_{n=0}^{\infty}\frac{\cos\left[\left(n+\frac{1}{2}\right)(x-\pi)\right]}{\sigma^2+\left(n+\frac{1}{2}\right)^2}, \qquad |x|\le\pi.$$

Verify his proof.

Step 1: Using the residue theorem, show that

$$\frac{1}{2\pi i}\oint_{C_N}\pi\frac{\sinh(xz)}{\cosh(\pi z)}\frac{dz}{z-\sigma} = \pi\frac{\sinh(\sigma x)}{\cosh(\sigma\pi)} - \sum_{n=-N-1}^{N}\frac{(-1)^n\sin\left[\left(n+\frac{1}{2}\right)x\right]}{\sigma-i\left(n+\frac{1}{2}\right)},$$

where C_N is a circular contour that includes the poles $z=\sigma$ and $z_n = \pm i\left(n+\frac{1}{2}\right)$, $n=0,1,2,\ldots,N$.

Step 2: Show that in the limit of $N\to\infty$, the contour integral vanishes. [Hint: Examine the behavior of $z\,\sinh(xz)/[(z-\sigma)\cosh(\pi z)]$ as $|z|\to\infty$. Use (1.9.7) where C_R is the circular contour.]

Step 3: Break the infinite series in Step 1 into two parts and simplify.

In the next chapter we shall show how we can obtain the same series by direct integration.

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[8] Strutt, M. J. O., 1934: Berechnung des hochfrequenten Feldes einer Kreiszylinderspule in einer konzentrischen leitenden Schirmhülle mit ebenen Deckeln. *Hochfrequenztechn. Elecktroak.*, **43**, 121–123.